Applications of Heun Functions and Regge Theory in Gravitational Wave Astrophysics

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Motivations

- Instrumentation, Theoretical and Computational advances in black hole astrophysics driven by the work of LIGO, EHT and many others

- Mathematical Objects ↔Astrophysical Observations leading to "Mathematical Astrophysics"

- Roots in Einstein's use of differential geometry to understand spacetime

- Several recent successes: 2-spinor formalism of Newman-Penrose[1], BMS Group[2], Christodolou's memory effect[3] and so on and so forth.

1.Newman, E. and Penrose, R., "An approach to gravitational radiation by a method of spin coefficients", J.Math.Phys.,1962.
2. Alessio, F. and Esposito, G., "On the structure and applications of the Bondi–Metzner–Sachs group", Int. J. Geom. Methods Mod. Phys.,2018
3. Thorne, K.S., "Gravitational-wave bursts with memory: The Christodoulou effect", Phys. Rev. D., 1992

Foundations of Formalism I: The Teukolsky Equation

- The Teukolsky (Master) Equation[4], introduced by Saul Teukolsky in 1973, governs the behaviour of all spin weighted fields in Kerr spacetime.

- Derived using the Newman-Penrose formalism, a suitable tetrad and some algebraic manipulations leading to a master equation for spacetime variables.

- The spin weights $|s| = 0, 1/2, 1$describe scalar, neutrino and electromagnetic fields whereas $|s| = 2$ describes the metric perturbations which leads to the emission of gravitational wave signals.

4. Teukolsky, S. A. "Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-Field Perturbations", ApJ, 1973

The Teukolsky Master Equation

- The Master equation in operator form can be written as:

$$\mathscr{L}_x^{(s)}\, {}_s\psi(x) = \Sigma\, {}_sT(x)$$

where, ${}_s\psi(x)$ is the Weyl Scalar for each spin weight $|s|$, $\mathscr{L}_x^{(s)}$ is the Teukolsky differential operator in Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$ is given by:

$$\mathscr{L}_x^{(s)} = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2} + \left(\frac{4Mar}{\Delta}\right)\frac{\partial^2}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\varphi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\varphi} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t} + (s^2\cot^2\theta - s)$$

and ${}_sT(x)$ is the source term built from the energy-momentum tensor.

- For black hole spin $a = 0$, it reduces to the Bardeen-Press equation that governs perturbations of Schwarzschild black hole.

Mode Ansatz and Decoupling

- The Master Equation can be separated by applying a mode separability ansatz that isolates contributions from isometries of Kerr arising from the temporal and polar Killing vectors:

$$ {}_s\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r) $$

Teukolsky Radial and Angular Equations

- For the source free case ${}_sT = 0$, by applying the mode ansatz we get the Teukolsky Radial Equation (TRE):

$$\Delta^{-s}\frac{d}{dr}\left[\Delta^{s+1}\frac{dR(r)}{dr}\right] + \left[\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right]R(r) = 0,$$

and the Teukolsky Angular Equation (TAE):

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + a^2\omega^2\cos^2\theta - 2sa\omega\cos\theta - \frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta} + {}_sA_{\ell m}(a\omega)\right]S(\theta) = 0$$

where, $K \equiv (r^2 + a^2\omega - am)$, $\lambda_m \equiv_s A_{lm} + a^2\omega^2 - 2am\omega - s(s+1)$ and ${}_sA_{lm}$ is the angular eigenvalue.

- Note that the equations are coupled in terms of ω and ${}_sA_{lm}$.

Teukolsky Radial and Angular Equations

- The TRE is used to calculate a large number of astrophysical observables such as fluxes from particles orbiting the black hole[5], quasinormal modes[6], behaviour of magnetic fields around current loops[7] and so on.

- The TAE corresponds to a class of functions called "spin weighted spheroidal harmonics"[8]. These are generalisations of the well known spin weighted *spherical* harmonics that are used in everything from the hydrogen atom to numerical relativity.

5.Drasco, S. and Hughes, S., "Gravitational wave snapshots of generic extreme mass ratio inspirals", Phys. Rev. D., 2006
6. Cook, G.B. and Zalutskiy, M., "Gravitational perturbations of the Kerr geometry: High-accuracy study", Phys. Rev. D., 2014
7.Linet, B., "Stationary axisymmetry electromagnetic fields in the Kerr metric", J. Phys. A: Math. Gen., 1979
8. Berti, E., Cardoso, V. and Casals, M. "Eigenvalues and eigenfunctions of spin-weighted spheroidal harmonics in four and higher dimensions", Phys. Rev. D., 2006

Foundations of Formalism II. Heun Equations

- The general Heun equation is the most general linear, second order differential equation with four regular singularities, including infinity[9].

- By a confluence/combining of two of its regular singularities into one we obtain the confluent Heun equation for the confluent Heun function H^c:

$$\frac{d^2H^c}{dz^2} + \left(4p + \frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{dH^c}{dz} + \frac{4\alpha pz - \sigma}{z(z-1)}H^c = 0$$

- The confluent Heun equation has two regular singularities ($z = 0,1$) and one irregular singularity at infinity.

- By repeating the confluence procedure, we can obtain bi-confluent and triconfluent Heun functions as well.

- Wide applications in everything from quantum computing, atomic physics and so on[9]

9. Slavyanov, S. and Lay, W. "Special Functions: A Unified Theory Based on Singularities" Oxford Mathematical Monographs, 2000

Teukolsky Equations and Confluent Heun equations

- By transformation of the independent variable and a suitable mapping of parameters[10], both TRE and TAE for Kerr can be mapped to confluent Heun equations (for Kerr de-Sitter we get the Heun equation[11]).

- **Therefore, both TRE and TAE represent the same mathematical object.**

- Thus the problem of solving the TRE and TAE boils down to obtaining solutions of confluent Heun functions

10. Borissov, R. And Fiziev, P.P., "Exact Solutions of Teukolsky Master Equation with Continuous Spectrum", Bulg. J Phys., 2010.
11. Suzuki, H.,, Takasugi,E. And Umetsu,H. "Perturbations of Kerr-de Sitter Black Hole and Heun's Equations", Prog. Theor. Phys., 1998.

Existing Techniques for Computation

- Leaver's method[12]: using coulomb functions leading to a continued fraction to be solved by root finding
- Mano-Suzuki-Takesugi method[13]: matched asymptotic expansion based on Gauss hypergeometric and coulomb spheroidal functions
- What can Heun's function accomplish?

12.Leaver, E.W., "An Analytic Representation for the Quasi-Normal Modes of Kerr Black Holes", Proc. Roy. Soc. Lond. A., 1985.
13. Mano, S. And Suzuki, H. And Takesugi, E.. "Analytical solutions of the Teukolsky Equation and their low frequency expansions", Prog. Theor. Phys., 1996.

The Confluent Heun form of the Radial and Angular Equations (joint work with Pierre-Louis Giscard)[14]

14. Giscard, P. And Tamar, A. "Elementary Integral Series for Heun Functions. With an Application to Black-Hole Perturbation Theory", arXiv:2010.03919, 2021

Computation using Heun Functions

- Turns out, quite a bit!
- In work with Pierre-Louis Giscard, using his technique of path sums, we were able to map the heun class of equations to an Volterra integral of second kind
- This integral contains as a solution the Neumann series which is an **unconditionally convergent series.**
- Moreover, the technique of path sums works for **any system of coupled linear differential equations with variable coefficients**
- This also resolved a long standing problem in mathematical literature of finding an integral representation **(<u>not a transform</u>**) of the Heun class of functions

Path Sum Method for Computation

Heun Equation

↓

Matrix Form with variable coefficients as the elements

↓

First Order Differential Equation for the Matrix

↓

Solution as a Path Ordered Exponential

↓

Evaluate using Graph Theory Techniques (Path-Sums)

↓

Volterra Integral of Second Kind

↓

Unconditionally Convergent Solution of a Neumann Series

↓

Numerically Efficient Implementation

Computation using Heun Functions

- For the TRE and TAE, which are confluent Heun equations this implied a solution that is **uniformly convergent from black hole horizon to spatial infinity**

- Serious alternative to the well used Mano-Suzuki-Takesugi formulation

- **Some features of the formulation:**

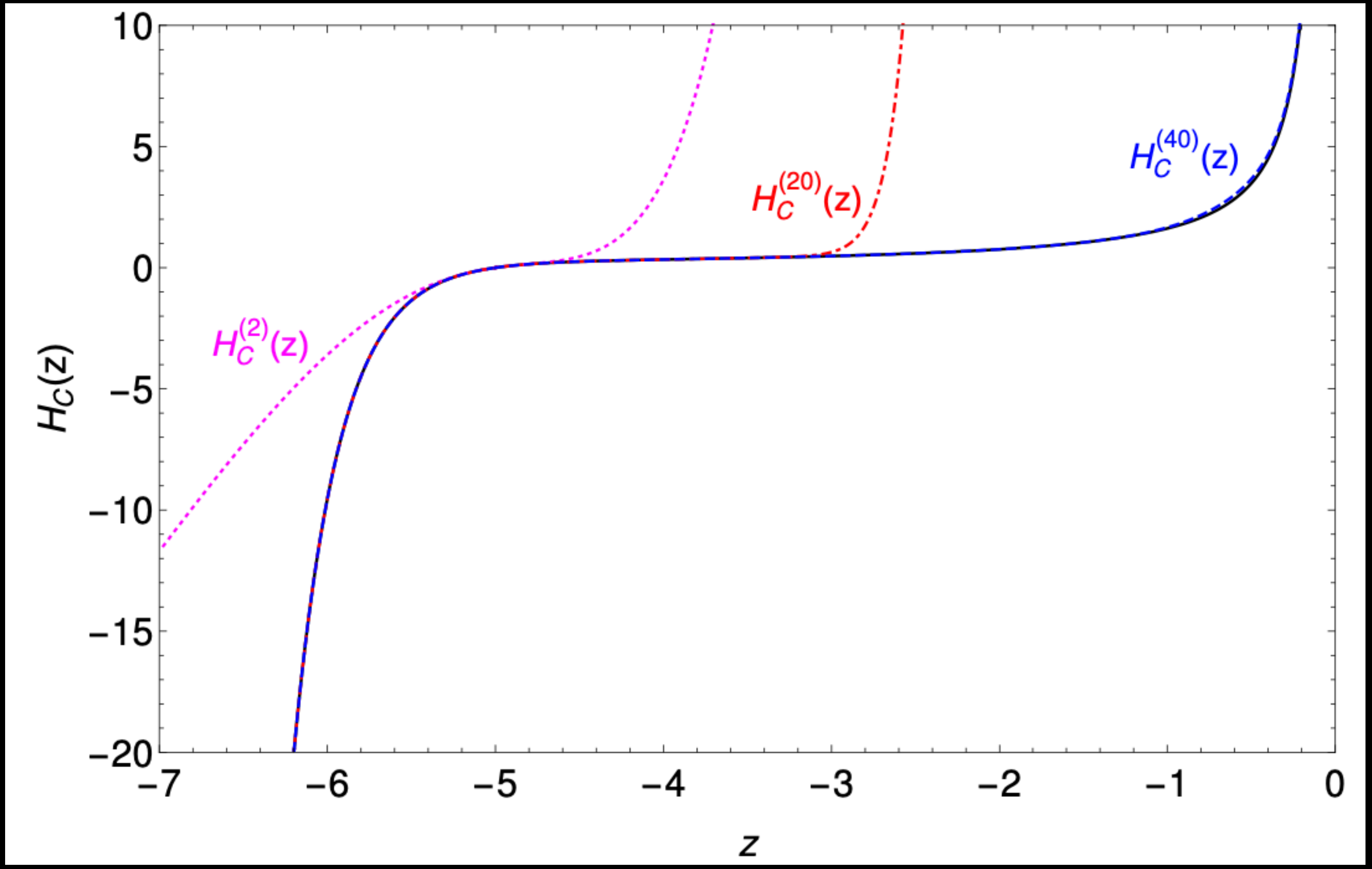
1. **The technique is numerically very efficient**

2. **It did not require introduction of any other special functions or an auxiliary parameter**

3. **It has no restriction on the parameters of the TRE/TAE (real or complex, high or low black hole spin, all spin fields etc)**

4. **It treats TRE and TAE on the same mathematical footing**

Results



Convergence near z = 0 is slowed down due to [one of the integrans] being singular at z = 0 just as [the confluent Heun equation] is. <u>Still, the integral series is convergent over the entire domain $z \in [0,\infty]$,</u> **a crucial property for perturbative black hole theory that is unique to the present approach.**

Code	N	Time (sec.)
HeunG	1000	2.22
Python	1000	0.0096
HeunG	10000	22.4
Python	10000	0.090
HeunG	50000	110.8
Python	50000	0.43
HeunG	100000	231
Python	100000	0.90
HeunG	200000	464
Python	200000	1.87

A comparative study for N sample points between Mathematica's HeunG function and of our implementation in Mathematica (presented at DoD,2021)

Current and Future Work

- Reducing computational complexities from Green's function of the TRE and TAE wherein the MST method leads to significant complications in quasinormal modes and late time decays[15]

- Numerically well behaved implementation of spin weighted spheroidal harmonics for both, real and complex frequency parameter <u>including their series summations (applications in wave scattering of Kerr Black Holes)</u>

- Circumvent difficulties from long range nature of the Teukolsky potential (avoiding the Sasaki Nakamura transform[16]).

- Conserved currents using integral symmetries of confluent Heun functions

- Mathematical Physics (Behaviour of confluent Heun functions near Stokes lines, asymptotic analysis etc.)

15.Casals, M. And Kavanagh, C. And Ottewill, A.C., "High-order late-time tail in a Kerr spacetime", Phys. Rev. D., 2016
16.Sasaki, M. And Nakamura, T. "Gravitational Radiation from a Kerr Black hole I. Formalism and a method Numerical Analysis", Prog. Theor. Phys., 1982

Complex Angular Momentum and Regge Theory in Black Hole Physics:

Quasinormal Modes of Kerr using Regge trajectories (joint work with Antoine Folacci)[17]

15. Folacci, A. And Tamar, A. "Quasinormal mode frequencies of Kerr black holes from Regge trajectories", arXiv:2103.01258, 2020

Complex Angular Momentum Techniques and Black Holes

- Goebel[19] in 1972 suggested that black hole normal could be interpreted as gravitational waves spiralling near the unstable photon orbit at $r = 3M$ of Schwarzschild and radiating energy.

- Chandrashekhar and Ferrari[20] used CAM techniques to study resonant behaviour of stars, by exploiting the theory of Regge Poles.

- Rigorous theory worked out by Folacci and collaborators over the past 2 decades for Schwarzschild Black holes, mainly in scattering of waves and correspondence between geodesics and resonant frequencies.

- Crudely, it is a different viewpoint of studying resonance excitations of a black hole wherein the correspondence to geodesics is much more transparent than the conventional quasinormal mode picture.

- Our work was the first application to Kerr for non-zero spin weight fields in full generality.

19.Goebel, C.J., "Comments on the "vibrations of a black hole", ApJ., 1972
20. Chandrashekhar, S. And Ferrari, V., "On the Non-Radial Oscillations of a Star: V. A Fully Relativistic Treatment of a Newtonian Star", Proc. Roy. Soc. Lond. A., 1995

Foundations of Formalism: Complex Angular Momentum Techniques

- First used by Watson to study diffraction of radio waves around the earth.
- Explosion in applications of particle physics due to the work of T. Regge. It helped classify stable and decaying particles.
- A notion of a "surface wave" that allows study of diffraction effects in scattering. Andersson interpreted these waves to be travelling close to the unstable photon orbit of Schwarzschild[18].

16.Andersson, N. And Thylwe, K.E., "Complex Angular Momentum approach to black hole scattering", Class. Quant. Grav., 1994

Model Implementation for Schwarzschild Black Holes

- Begin with the equation governing the perturbations of the Black Holes

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_l(r)\right]\phi_{\omega l} = 0$$

- Impose the physical boundary conditions of being purely ingoing at black hole horizon and purely outgoing at infinity

$$\phi_{\omega l}(r) \sim_{r_* \to -\infty} e^{-i\omega r_*},$$

$$\phi_{\omega l}(r) \sim_{r_* \to \infty} A_l^-(\omega)e^{-i\omega r_*} + A_l^+(\omega)e^{i\omega r_*} \ ,$$

where $A_l(\omega)$ are complex.

- The Scattering (S)- Matrix Elements are given by:

$$S_l(\omega) = e^{i(l+1)\pi}\frac{A_l^+(\omega)}{A_l^-(\omega)}$$

Model Implementation for Schwarzschild Black Holes

- The S-matrix elements are substituted into the the scattering amplitudes $f^+(\omega, x)$ and $f^-(\omega, x)$ which are given by:

$$f^{\pm}(\omega, x) = \mathscr{L}^{\pm}\tilde{f}^{\pm}(\omega, x)$$

where,

$$\tilde{f}^{\pm}(\omega, x) = \frac{1}{2i\omega}\sum_{l=2}^{\infty}\frac{(2l+1)}{(l-1)l(l+1)(l+2)}\left[\frac{1}{2}\left(S_l^{(e)} \pm S_l^{(o)}\right) - \left(\frac{1 \pm 1}{2}\right)\right]P_l(x)$$

having, $\mathscr{L}$ as the operator that has Legendre polynomials $P_l(x)$ as its eigenfunctions and $S_l^{(e,o)}$ correspond to scattering matrix elements corresponding to even and odd parity modes

Model Implementation for Schwarzschild Black Holes

- Analytically replace the discrete summation over angular momentum parameter l by a contour integral in the complex $\lambda = l + 1/2$ plane (i.e "complex angular momentum" plane) by using the **Sommerfeld-Watson transform:**

$$\sum_{l=2}^{\infty} (-1)^l F(l) = \frac{i}{2} \int_{C'} \frac{F(\lambda - 1/2)}{\cos(\pi\lambda)}$$

which holds for a function F without any singularities on the real λ axis.

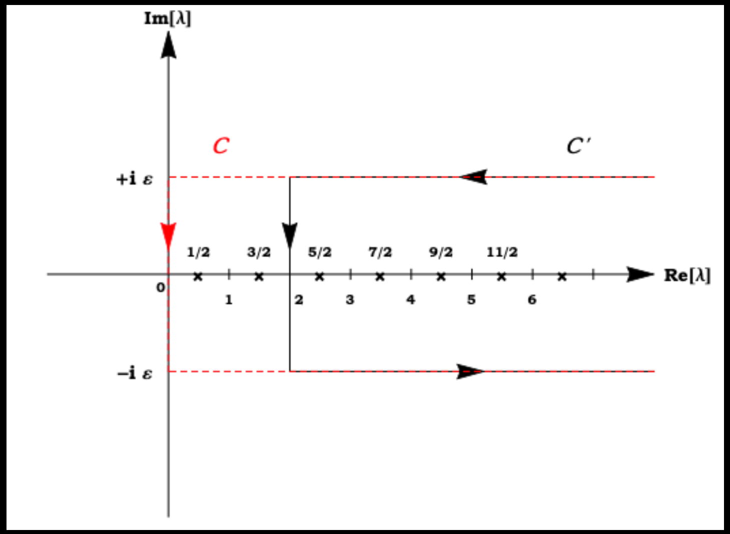
- Analytically extend the Legendre polynomials $P_l(x)$ to a complex index $P_{\lambda-1/2}$ using the formula:

$$P_{\lambda-1/2}(x) =_2 F_1[1/2 - \lambda, 1/2 + \lambda; 1; (1 - z)/2]$$

- Extend the S-matrix elements to complex values as well

Model Implementation for Schwarzschild Black Holes

- To collect contributions of Regge poles, deform the contour from C' to C:



- Remove contributions arising from the two spurious poles introduced above.

- Deform the contour, and by **using Cauchy's theorem, collect the Regge pole contributions** and discard the solutions of background integrals and quarter circles at infinity (valid in "high energy" approximation

- The resultant formula is a sum over Regge Poles lying in the first quadrant of the CAM plane involving the residues of the matrices of the S-matrix element at these poles

Main Takeaway

- Apply the Sommerfeld-Watson transform to the scattering amplitude. **This converts the summation over a real parameter l to an integral in the complex plane for parameter $\lambda - 1/2$.**

- Deform the contour of integration to get singularities. These correspond to singularities of S-matrix in the CAM plane. **These are known as Regge poles $\lambda_n(\omega)$.** Cauchy's theorem can be used to collect contributions of Regge poles.

- **As ω (which is now real)** varies, Regge Poles trace out a path in the CAM plane called Regge trajectories.

- Whenever the real part of the pole intersects with a half-integer, we have a resonance.

- The Breit-Wigner formula gives us the real and complex frequencies which are our QNMs.

- For reference see Decanini, Folacci, Jensen et. al.[21]

17. Decanini, Y. And Folacci, A. And Jensen, B. "Complex angular momentum in black hole physics and quasinormal modes", Phys. Rev. D., 2003

Regge Theory for Kerr Black Holes

- In joint work with Antoine Folacci, we attempted to study the QNMs of Kerr spacetime using CAM techniques.
- We focused on the Green's function of the Teukolsky Equation.
- Our main aim was to improve the correspondence between QNMs and geodesics, generalising existing studies of Regge Poles as well Green's function of Kerr for non-zero spin weighted fields.

Regge Theory for Kerr Black Holes

- Fundamental difference from the Schwarzschild case: We now have summation over **two** indices (l, m) of our functions.

- We only considered the Sommerfeld-Watson transform for the usual complex "angular momentum" (and still obtained highly accurate numerical results)

- However a complete treatment requires studying the functions in several complex variables (scope for future work!)

Foundations of Formalism: Green's Function of the Teukolsky Equation

- We study the retarded Green's function ${}_sG_{\text{ret}}(x, x')$ solution of the equation

$$\mathscr{L}_x^{(s)}\, {}_sG_{\text{ret}}(x, x') = \Sigma\, \delta^{(4)}(x, x'), \text{ (here: } \Sigma = r^2 + a^2 \cos^2 \theta)$$

where,

$${}_sG(x, x') = -\frac{(\Delta')^s}{2\pi} \sum_{\ell=|s|}^{\infty} \sum_{m=-\ell}^{+\ell} \int_{-\infty+ic}^{+\infty+ic} d\omega\, e^{-i\omega t + im\varphi}\, {}_sG_{\ell m}(r, r'; \omega)\, {}_sS_{\ell m}(\theta, a\omega)\, {}_sS^*_{lm}(\theta', a\omega)$$

having, $\Delta = r^2 - 2Mr + a^2$, $R(r), S(\theta)$ being solutions of the TRE and TAE with suitable boundary conditions. The radial contribution is given by:

$${}_sG_{\ell m}(r, r'; \omega) = \frac{{}_sR^{\text{in}}_{\ell m}(r_<, \omega)\, {}_sR^{\text{up}}_{\ell m}(r_>, \omega)}{{}_sW_{\ell m}(\omega)},$$

and the Wronskian ${}_sW_{\ell m}(\omega)$ is given by:

$${}_sW_{\ell m}(\omega) = \Delta^{s+1} \left({}_sR^{\text{in}}_{\ell m}(r, \omega) \frac{d}{dr}\, {}_sR^{\text{up}}_{\ell m}(r, \omega) - {}_sR^{\text{up}}_{\ell m}(r, \omega) \frac{d}{dr}\, {}_sR^{\text{in}}_{\ell m}(r, \omega) \right)$$

When evaluated at the suitable boundary conditions gives us:

$${}_sW_{lm}(\omega) = (2i\omega)\mathscr{A}_{lm}(\omega)$$

where, $A_{lm}(\omega)$ like for Schwarzschild is one of the coefficients obtained from specifying the asymptotic behaviour at the boundary conditions

- All other notation is the same as the Teukolsky equation.

Application of the Regge Theory

- Follow the same prescription as the one discussed in the primer.
- Obtained quasinormal modes as poles of the Wronskian, under the (well-behaved) assumption of

$$Re(\omega_{lmn}) >> Im(\omega_{lmn})$$

- This "high frequency" limit discards the contributions of the background integrals arising in the Regge theory.

Application of the Regge Theory

- The resultant expression of the Green's function after applying Regge theory is as follows:

$$ {}_sG^{RP}(x,x') = i(\Delta')^s \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{+\infty} \int_{-\infty+ic}^{+\infty+ic} d\omega e^{-i\omega t + im\varphi} \left(\frac{{}_s r_{nm}(\omega)}{4\omega} \right) \frac{e^{i\pi[\lambda_{nm}(\omega)-1/2]}}{\cos[\pi\lambda_{nm}(\omega)]} {}_sR^{\text{in}}_{\lambda_{nm}(\omega)-1/2,m}(r_<,\omega)\, {}_sR^{\text{up}}_{\lambda_{nm}(\omega)-1/2,m}(r_>,\omega)\, {}_sS_{\lambda_{nm}(\omega)-1/2,m}(\theta,a\omega)\, {}_sS^*_{\lambda_{nm}(\omega)-1/2,m}(\theta',a\omega) $$

where, ${}_sr_{nm}(\omega)$ contains the residue contributions which are in turn obtained from asserting zeroes of the Wronskian and $\lambda_{nm}(\omega)$ are the Regge Poles.

- Note: Recalling that the Radial and Angular Functions $R(r), S(\theta)$ are equivalently confluent Heun functions, this is the Green's function for the confluent Heun functions!

Results: Overview

- Recovered all qualitative features of QNMs of Kerr:

1. splitting of “azimuthal degeneracy” in QNMs of Schwarzschild leading to a characteristic pattern of Kerr QNMs

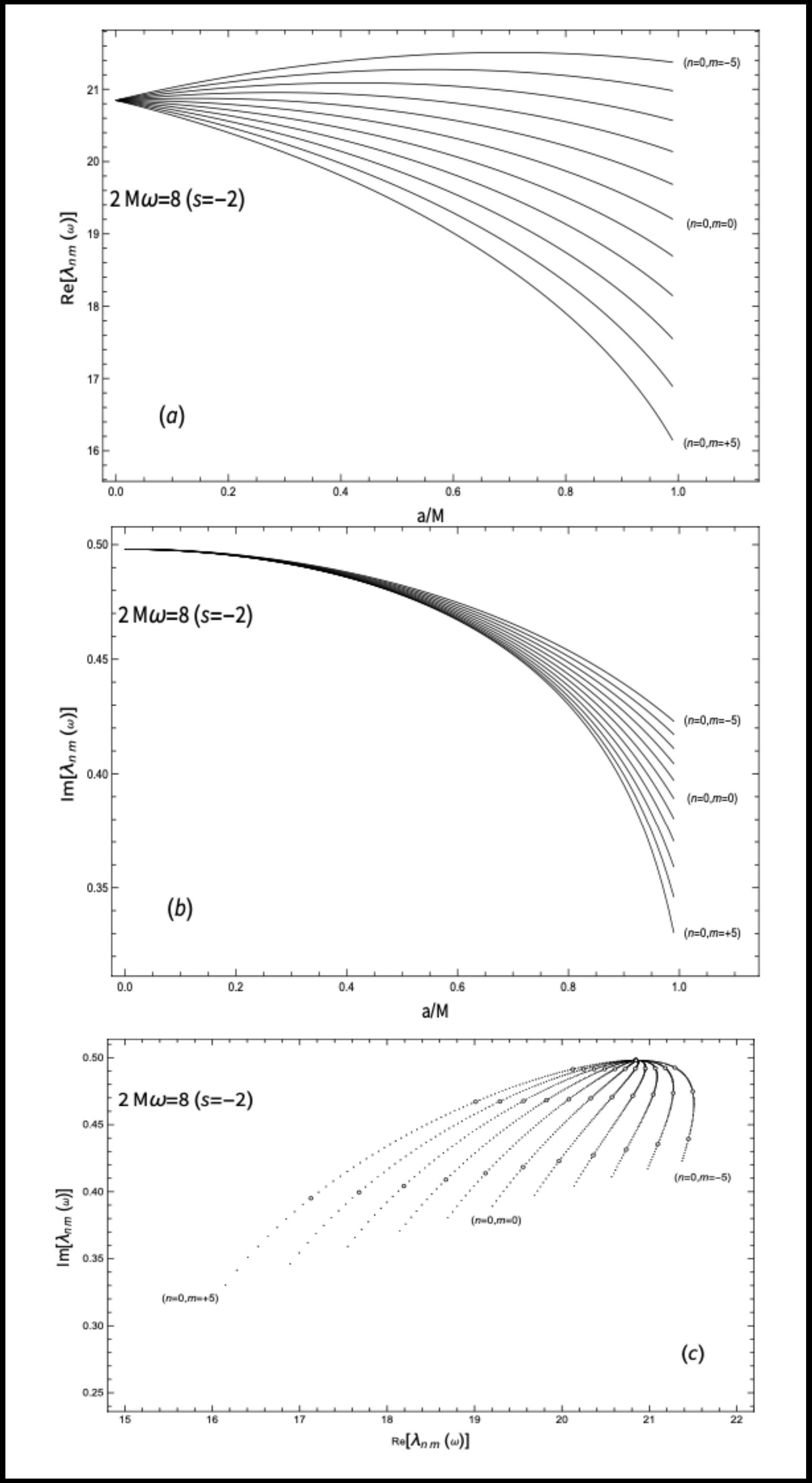
2. Reasonably asymptotic to algebraically special frequencies for near extremal cases

3. **For the astrophysically relevant $|s| = 2$, numerical accuracy ranging from $10^{-1} - 10^{-3}$ for low frequency and $10^{-5} - 10^{-8}$ for high frequency case.**

4. **Valid for the highly relevant $l = m = 2, n = 0$ fundamental mode** as well for subdominant modes (thus suitable for multimode analysis and studying sub-dominant contributions, see work by Giesler[22] et. al.)

18. Giesler, M.,Isi. M.,Scheel, M.A.,Teukolsky,S.”Black Hole ringdown-the importance of overtones”,Phys. Rev. X., 2019

Results: Sample Plot



Demonstrates the characteristic “splitting” of the QNMs as the spin increases (Gravitational analogue of the Zeeman effect)

Results: Tabulation

Results for $m = 0, n = 0$

ℓ	a/M	$2M\omega_{\ell 00}$ (exact)	$2M\omega_{\ell 00}$ (semiclassical)
2	0	$0.74734337 - 0.17792463i$	$0.75812003 - 0.17620450i$ $(-1.4 \times 10^{-2}, 9.7 \times 10^{-3})$
	0.3	$0.75397012 - 0.17670656i$	$0.7645048 - 0.17509216i$ $(-1.4 \times 10^{-2}, 9.1 \times 10^{-3})$
	0.6	$0.77610784 - 0.17198934i$	$0.78581861 - 0.17066938i$ $(-1.3 \times 10^{-2}, 7.7 \times 10^{-3})$
	0.9	$0.82400893 - 0.15696539i$	$0.83203054 - 0.1562239i$ $(-9.7 \times 10^{-3}, 4.7 \times 10^{-3})$
3	0	$1.19888658 - 0.18540610i$	$1.20332642 - 0.18459133i$ $(-3.7 \times 10^{-3}, 4.4 \times 10^{-3})$
	0.3	$1.20763994 - 0.18415232i$	$1.21198465 - 0.18337758i$ $(-3.6 \times 10^{-3}, 4.2 \times 10^{-3})$
	0.6	$1.23655149 - 0.17938208i$	$1.24058739 - 0.17867688i$ $(-3.3 \times 10^{-3}, 3.9 \times 10^{-3})$
	0.9	$1.29734769 - 0.1650988i$	$1.30082362 - 0.16456715i$ $(-2.7 \times 10^{-3}, 3.2 \times 10^{-3})$
4	0	$1.61835676 - 0.18832792i$	$1.62051635 - 0.18797143i$ $(-1.3 \times 10^{-3}, 1.9 \times 10^{-3})$
	0.3	$1.62929183 - 0.18704894i$	$1.63139858 - 0.18670232i$ $(-1.3 \times 10^{-3}, 1.9 \times 10^{-3})$
	0.6	$1.66528361 - 0.18223275i$	$1.66722321 - 0.18192134i$ $(-1.1 \times 10^{-3}, 1.7 \times 10^{-3})$
	0.9	$1.74039368 - 0.16821049i$	$1.74204608 - 0.16797157i$ $(-9.5 \times 10^{-4}, 1.4 \times 10^{-3})$
5	0	$2.02459062 - 0.18974103i$	$2.02578979 - 0.18956947i$ $(-5.9 \times 10^{-4}, 9.0 \times 10^{-4})$
	0.3	$2.03772958 - 0.18844408i$	$2.03889667 - 0.18827929i$ $(-5.7 \times 10^{-4}, 8.7 \times 10^{-4})$
	0.6	$2.08091194 - 0.18358986i$	$2.08197908 - 0.18344489i$ $(-5.1 \times 10^{-4}, 7.9 \times 10^{-4})$
	0.9	$2.17076254 - 0.16967079i$	$2.17166331 - 0.16956021i$ $(-4.1 \times 10^{-4}, 6.5 \times 10^{-4})$
6	0	$2.42401964 - 0.19053169i$	$2.42475123 - 0.19043973i$ $(-3.0 \times 10^{-4}, 4.8 \times 10^{-4})$
	0.3	$2.43938141 - 0.18922259i$	$2.4400923 - 0.18913122i$ $(-2.9 \times 10^{-4}, 4.8 \times 10^{-4})$
	0.6	$2.48983218 - 0.18434151i$	$2.49047917 - 0.18426280i$ $(-2.6 \times 10^{-4}, 4.3 \times 10^{-4})$
	0.9	$2.59466026 - 0.17047252i$	$2.59520293 - 0.17041291i$ $(-2.1 \times 10^{-4}, 3.5 \times 10^{-4})$
7	0	$2.81947024 - 0.19101926i$	$2.81994843 - 0.19096663i$ $(-1.7 \times 10^{-4}, 2.8 \times 10^{-4})$
	0.3	$2.83706937 - 0.18970184i$	$2.83753351 - 0.18965010i$ $(-1.6 \times 10^{-4}, 2.7 \times 10^{-4})$
	0.6	$2.89484400 - 0.18480196i$	$2.89526508 - 0.18475609i$ $(-1.5 \times 10^{-4}, 2.5 \times 10^{-4})$
	0.9	$3.01479937 - 0.17096099i$	$3.01515102 - 0.17092672i$ $(-1.2 \times 10^{-4}, 2.0 \times 10^{-4})$
30	0	$11.72303056 - 0.19236532i$	$11.72303774 - 0.19236509i$ $(-6.1 \times 10^{-7}, 1.2 \times 10^{-6})$
	0.3	$11.79304319 - 0.19102129i$	$11.79305013 - 0.19102106i$ $(-5.9 \times 10^{-7}, 1.2 \times 10^{-6})$
	0.6	$12.02265939 - 0.18605995i$	$12.02266563 - 0.18605978i$ $(-5.2 \times 10^{-7}, 9.1 \times 10^{-7})$
	0.9	$12.49864059 - 0.17228421i$	$12.49864573 - 0.17228409i$ $(-4.1 \times 10^{-7}, 7.0 \times 10^{-7})$
50	0	$19.42754200 - 0.19241919i$	$19.42754358 - 0.19241917i$ $(-8.1 \times 10^{-8}, 1.0 \times 10^{-7})$
	0.3	$19.54335348 - 0.19107398i$	$19.54335501 - 0.19107395i$ $(-7.8 \times 10^{-8}, 1.6 \times 10^{-7})$
	0.6	$19.92316116 - 0.18610987i$	$19.92316253 - 0.18610984i$ $(-6.9 \times 10^{-8}, 1.6 \times 10^{-7})$
	0.9	$20.71044316 - 0.17233638i$	$20.71044426 - 0.17233634i$ $(-5.3 \times 10^{-8}, 2.3 \times 10^{-7})$

Results for $m = 2, n = 0$

ℓ	a/M	$2M\omega_{\ell 20}$ (exact)	$2M\omega_{\ell 20}$ (semiclassical)
2	0	$0.74734337 - 0.17792463i$	$0.75812003 - 0.17620450i$ $(-1.4 \times 10^{-2}, 9.7 \times 10^{-3})$
	0.3	$0.83905336 - 0.17545854i$	$0.84858806 - 0.17476829i$ $(-1.1 \times 10^{-2}, 3.9 \times 10^{-3})$
	0.6	$0.98808956 - 0.1675304i$	$0.99400184 - 0.16816867i$ $(6.0 \times 10^{-3}, -3.8 \times 10^{-3})$
	0.9	$1.34322854 - 0.12973847i$	$1.34646762 - 0.11997547i$ $(-2.4 \times 10^{-3}, 7.5 \times 10^{-2})$
3	0	$1.19888658 - 0.18540610i$	$1.20332642 - 0.18459133i$ $(-3.7 \times 10^{-3}, 4.4 \times 10^{-3})$
	0.3	$1.29591615 - 0.18304959i$	$1.30058588 - 0.18228513i$ $(-3.6 \times 10^{-3}, 4.2 \times 10^{-3})$
	0.6	$1.44560394 - 0.17455992i$	$1.45006400 - 0.17399203i$ $(-3.1 \times 10^{-3}, 3.3 \times 10^{-3})$
	0.9	$1.75236564 - 0.13780946i$	$1.75559843 - 0.13795719i$ $(-1.8 \times 10^{-3}, -1.1 \times 10^{-3})$
4	0	$1.61835676 - 0.18832792i$	$1.62051635 - 0.18797143i$ $(-1.3 \times 10^{-3}, 1.9 \times 10^{-3})$
	0.3	$1.71943307 - 0.18637371i$	$1.72174745 - 0.18600639i$ $(-1.3 \times 10^{-3}, 2.0 \times 10^{-3})$
	0.6	$1.87374023 - 0.17867501i$	$1.87609617 - 0.17833697i$ $(-1.3 \times 10^{-3}, 1.9 \times 10^{-3})$
	0.9	$2.16717156 - 0.14661018i$	$2.16981334 - 0.14646594i$ $(-1.2 \times 10^{-3}, 9.8 \times 10^{-4})$
5	0	$2.02459062 - 0.18974103i$	$2.02578979 - 0.18956947i$ $(-5.9 \times 10^{-4}, 9.0 \times 10^{-4})$
	0.3	$2.12876078 - 0.18799282i$	$2.13004041 - 0.18780611i$ $(-6.0 \times 10^{-4}, 9.9 \times 10^{-4})$
	0.6	$2.28843643 - 0.18084652i$	$2.28976627 - 0.18067732i$ $(-5.8 \times 10^{-4}, 9.4 \times 10^{-4})$
	0.9	$2.58114884 - 0.15252946i$	$2.58295260 - 0.15241320i$ $(-7.0 \times 10^{-4}, 7.6 \times 10^{-4})$
6	0	$2.42401964 - 0.19053169i$	$2.42475123 - 0.19043973i$ $(-3.0 \times 10^{-4}, 4.8 \times 10^{-4})$
	0.3	$2.53084884 - 0.18889435i$	$2.5316235 - 0.18879687i$ $(-3.1 \times 10^{-4}, 5.2 \times 10^{-4})$
	0.6	$2.69635931 - 0.18211885i$	$2.69717704 - 0.18202535i$ $(-3.0 \times 10^{-4}, 5.1 \times 10^{-4})$
	0.9	$2.99423407 - 0.1564351i$	$2.99548730 - 0.15635765i$ $(-4.2 \times 10^{-4}, 5.0 \times 10^{-4})$
7	0	$2.81947024 - 0.19101926i$	$2.81994843 - 0.19096663i$ $(-1.7 \times 10^{-4}, 2.8 \times 10^{-4})$
	0.3	$2.92875446 - 0.18944752i$	$2.92925719 - 0.18939035i$ $(-1.7 \times 10^{-4}, 3.0 \times 10^{-4})$
	0.6	$3.10044539 - 0.18293477i$	$3.10098473 - 0.18288259i$ $(-1.7 \times 10^{-4}, 2.9 \times 10^{-4})$
	0.9	$3.40668526 - 0.15913131i$	$3.40759035 - 0.15907999i$ $(-2.7 \times 10^{-4}, 3.2 \times 10^{-4})$
30	0	$11.72303056 - 0.19236532i$	$11.72303774 - 0.19236509i$ $(-6.1 \times 10^{-7}, 1.2 \times 10^{-6})$
	0.3	$11.88447941 - 0.19098059i$	$11.88448697 - 0.19098038i$ $(-6.4 \times 10^{-7}, 1.1 \times 10^{-6})$
	0.6	$12.22141596 - 0.18565394i$	$12.22142800 - 0.18565368i$ $(-9.9 \times 10^{-7}, 1.4 \times 10^{-6})$
	0.9	$12.85522429 - 0.16980436i$	$12.85526273 - 0.16980380i$ $(-3.0 \times 10^{-6}, 3.3 \times 10^{-6})$
50	0	$19.42754200 - 0.19241919i$	$19.42754358 - 0.19241917i$ $(-8.1 \times 10^{-8}, 1.0 \times 10^{-7})$
	0.3	$19.63459827 - 0.19104999i$	$19.63460002 - 0.19104996i$ $(-8.9 \times 10^{-8}, 1.6 \times 10^{-7})$
	0.6	$20.12077539 - 0.18586732i$	$20.12077891 - 0.18586728i$ $(-1.7 \times 10^{-7}, 6.5 \times 10^{-7})$
	0.9	$21.06270368 - 0.17087021i$	$21.06271683 - 0.17087010i$ $(-6.2 \times 10^{-7}, 6.4 \times 10^{-7})$

Future Avenues

- The precise mapping between QNMs and geodesics still remains unresolved. Need of new mathematical constructions to supplement the physical picture?
- Classification scheme for QNMs?
- Applications to Black Hole Imaging: Light Correlation Functions encoded in electric fields (see Chesler et. al.)[23]

23. Chessler, et. al., Light echos and coherent autocorrelations in a black hole spacetime, Class. Quant. Grav., 2021

"When one does a theoretical calculation, there are two ways of doing it: either one should have a clear physical model in mind or a rigorous mathematical basis."

–Enrico Fermi

Why not both?